

Weak Linear Mappings – A Survey

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There are various concepts of *structure preserving mappings* in geometry. It is the aim of the present paper to give a survey on geometrical characterizations of some of those mappings. We discuss the results for projective spaces in some detail and report on generalizations to other spaces.

We shall come across *partially defined mappings*. The notation $\varphi : \mathcal{P} \rightsquigarrow \mathcal{P}'$ is used in order to point out that φ is a mapping with *domain* $\text{dom } \varphi \subset \mathcal{P}$ and *image set* $\text{im } \varphi \subset \mathcal{P}'$. The *exceptional set* of φ (with respect to \mathcal{P}) is given as $\text{ex } \varphi := \mathcal{P} \setminus \text{dom } \varphi$. The mapping φ is said to be *globally defined* (with respect to \mathcal{P}) provided that $\text{ex } \varphi$ is empty. The notation $\varphi : \mathcal{P} \rightarrow \mathcal{P}'$ is maintained for globally defined mappings only.

If we are given any subset $\mathcal{M} \subset \mathcal{P}$, then $\{X^\varphi \mid X \in \mathcal{M} \cap \text{dom } \varphi\}$ is a well-defined set. By abuse of notation, it is written as \mathcal{M}^φ . Hence $\mathcal{M}^\varphi = \emptyset$ exactly for $\mathcal{M} \subset \text{ex } \varphi$.

1 Projective Spaces

1.1 Weak Semilinear Mappings of Vector Spaces

Throughout this paper \mathbf{V}, \mathbf{V}' denote right vector spaces over (not necessarily commutative) fields K, K' , respectively. We generalize the well-known concept of a semilinear mapping as follows: A *weak semilinear mapping*¹ $f : \mathbf{V} \rightarrow \mathbf{V}'$ with respect to a mapping $\zeta : K \rightarrow K'$ is additive and ζ -homogeneous, i.e.

$$(\mathbf{a} + \mathbf{b})^f = \mathbf{a}^f + \mathbf{b}^f \text{ and } (\mathbf{a}x)^f = \mathbf{a}^f x^\zeta \text{ for all } \mathbf{a}, \mathbf{b} \in \mathbf{V}, x \in K.$$

If $f \neq 0$, then ζ is a monomorphism which is uniquely determined by f .

Given such an $f \neq 0$ there exists a right vector space \mathbf{W} over $F := K^\zeta$ and a semilinear bijection $g : \mathbf{V} \rightarrow \mathbf{W}$ with respect to ζ (regarded as isomorphism $K \rightarrow F$). By the universal property of the tensor product $\mathbf{W} \otimes_F K'$, there is a K' -linear mapping $h : \mathbf{W} \otimes_F K' \rightarrow \mathbf{V}'$ such that $\mathbf{a}^f = (\mathbf{a}^g \otimes 1)^h$ for all $\mathbf{a} \in \mathbf{V}$.

The one- and two-dimensional subspaces of \mathbf{V} are the points and lines of the projective space on \mathbf{V} which is denoted by $(\mathcal{P}(\mathbf{V}), \mathcal{L}(\mathbf{V}))$. More generally, we put $\mathcal{P}(\mathbf{M}) := \{\mathbf{a}K \mid \mathbf{a} \in \mathbf{M} \setminus \{0\}\}$ for $\mathbf{M} \subset \mathbf{V}$; $\mathcal{P}(\mathbf{W})$ etc. is defined likewise. Each weak semilinear mapping $f : \mathbf{V} \rightarrow \mathbf{V}'$ determines a mapping of points

$$\varphi : \mathcal{P}(\mathbf{V}) \rightsquigarrow \mathcal{P}(\mathbf{V}'), \mathbf{a}K \mapsto (\mathbf{a}^f)K' \text{ for all } \mathbf{a}K \in \mathcal{P}(\mathbf{V} \setminus \ker f). \quad (1)$$

¹Such a mapping should not be mixed up with a *generalized semilinear mapping* [42] which yields a *homomorphism* of projective spaces.

The exceptional set of φ is the subspace $\mathcal{P}(\ker f)$. It follows that φ has the property

$$(\mathcal{X} \vee \mathcal{Y})^\varphi \subset \text{span}(\mathcal{X}^\varphi) \vee \text{span}(\mathcal{Y}^\varphi) \text{ for all subspaces } \mathcal{X}, \mathcal{Y} \subset \mathcal{P}(\mathbf{V}). \quad (2)$$

We remark that φ -images of subspaces need not be subspaces.

Suppose now that ζ is bijective, whence f is semilinear. If $\mathcal{X} \subset \mathcal{P}$ is a subspace, then so is \mathcal{X}^φ . Moreover, (2) improves to

$$(\mathcal{X} \vee \mathcal{Y})^\varphi = \mathcal{X}^\varphi \vee \mathcal{Y}^\varphi \text{ for all subspaces } \mathcal{X}, \mathcal{Y} \subset \mathcal{P}(\mathbf{V}). \quad (3)$$

1.2 Definition and Examples of Linear Mappings

In the sequel $(\mathcal{P}, \mathcal{L})$ and $(\mathcal{P}', \mathcal{L}')$ denote arbitrary projective spaces. Property (3) is adopted in the following purely geometric definition:

A *linear mapping*² of projective spaces is a mapping $\varphi : \mathcal{P} \rightarrow \mathcal{P}'$ (defined on some subset of \mathcal{P}) satisfying the following conditions:

- (L1) $(\{X\} \vee \{Y\})^\varphi = \{X\}^\varphi \vee \{Y\}^\varphi$ for all $X, Y \in \mathcal{P}$, $X \neq Y$.
- (L2) If $X^\varphi = Y^\varphi$ for distinct points $X, Y \in \text{dom } \varphi$, then there is at least one exceptional point on the line $\{X\} \vee \{Y\}$.

Condition (L2) is only used to rule out the possibility that *all* points of a line are mapped to the same point. There are three possibilities for the image of a line $l \in \mathcal{L}$:

- $\#(l \cap \text{ex } \varphi) \geq 2$: Then $l \subset \text{ex } \varphi$, i.e. we have an *exceptional line*.
- $\#(l \cap \text{ex } \varphi) = 1$: Then all points of $l \cap \text{dom } \varphi$ are mapped to the same point.
- $\#(l \cap \text{ex } \varphi) = 0$: Then $\varphi|_l$ is injective and $l^\varphi \in \mathcal{L}'$.

We give some examples of linear mappings:

1. The mapping (1) is linear, if $f : \mathbf{V} \rightarrow \mathbf{V}'$ is semilinear.
2. Let $\mathcal{S} \subset \mathcal{P}$ be a subspace. Write \mathcal{P}/\mathcal{S} for the set of all subspaces of the form $\mathcal{S} \vee \{X\}$, where $X \in \mathcal{P} \setminus \mathcal{S}$. Then \mathcal{P}/\mathcal{S} is the set of points of the quotient space of $(\mathcal{P}, \mathcal{L})$ modulo \mathcal{S} . The *canonical projection*

$$\psi_{\mathcal{S}} : \mathcal{P} \rightarrow \mathcal{P}/\mathcal{S}, \quad X \mapsto \mathcal{S} \vee \{X\} \text{ for all } X \in \mathcal{P} \setminus \mathcal{S}$$

is a surjective linear mapping with $\text{ex } \psi_{\mathcal{S}} = \mathcal{S}$.

3. With \mathcal{S} as above, the *canonical injection* $\iota_{\mathcal{S}} : \mathcal{S} \rightarrow \mathcal{P}$, $X \mapsto X$ is a globally defined linear mapping of the projective space on \mathcal{S} in \mathcal{P} .
4. A *projection* is based on two complementary subspaces of \mathcal{S}, \mathcal{T} of $(\mathcal{P}, \mathcal{L})$ as follows:

$$\pi : \mathcal{P} \rightarrow \mathcal{T}, \quad X \mapsto X^\pi \text{ with } \{X^\pi\} := (\mathcal{S} \vee \{X\}) \cap \mathcal{T} \text{ for all } X \in \mathcal{P} \setminus \mathcal{S}.$$

This π is a surjective linear mapping with $\text{ex } \pi = \mathcal{S}$.

5. Any collineation of projective spaces is a globally defined bijective linear mapping.

²This name has been used in Descriptive Geometry for more than 100 years. Some authors use other terminologies.

6. Suppose that $\varphi : \mathcal{P} \rightarrow \mathcal{P}'$ and $\varphi' : \mathcal{P}'' \rightarrow \mathcal{P}'''$ are linear mappings with $\text{im } \varphi \subset \mathcal{P}''$. Then $\varphi\varphi' : \mathcal{P} \rightarrow \mathcal{P}'''$ is also a linear mapping (possibly an empty mapping).
7. Let $\varphi : \mathcal{P} \rightarrow \mathcal{P}'$ be linear. The hyperplanes of $(\mathcal{P}, \mathcal{L})$ and $(\mathcal{P}', \mathcal{L}')$ are, respectively, the points of the dual projective spaces $(\mathcal{P}^*, \mathcal{L}^*)$ and $(\mathcal{P}'^*, \mathcal{L}'^*)$. The *extended* φ -preimage of any hyperplane $\mathcal{H}' \subset \mathcal{P}'$ is given by $\text{ex } \varphi \cup \mathcal{H}'^{\varphi^{-1}}$. Define

$$\varphi^T : \mathcal{P}'^* \rightarrow \mathcal{P}^*, \mathcal{H}' \mapsto \text{ex } \varphi \cup \mathcal{H}'^{\varphi^{-1}} \text{ for all } \mathcal{H}' \in \mathcal{P}'^* \text{ with } \text{im } \varphi \not\subset \mathcal{H}'$$

This *transpose mapping* of φ is linear [23], [29].

1.3 Brauner's Theorem

Theorem 1 (BRAUNER [7]) *Let $\varphi : \mathcal{P} \rightarrow \mathcal{P}'$ be a mapping of projective spaces.*

1. *If φ satisfies (L1), then the exceptional subset $\text{ex } \varphi \subset \mathcal{P}$ and the image set $\text{im } \varphi \subset \mathcal{P}'$ are subspaces.*
2. *If φ satisfies (L1) and $\#\text{im } \varphi \neq 1$, then φ is linear.*
3. *Each linear mapping φ is decomposable into the canonical projection $\mathcal{P} \rightarrow \mathcal{P}/\text{ex } \varphi$, a collineation of this quotient space onto the subspace $\text{im } \varphi$, and the canonical injection of $\text{im } \varphi$ in \mathcal{P}' .*

As a matter of fact BRAUNER did not discuss mappings φ satisfying (L1) and $\#\text{im } \varphi = 1$. It is immediate that here $\text{ex } \varphi$ can be any subspace of $(\mathcal{P}, \mathcal{L})$ other than \mathcal{P} .

By the second fundamental theorem of projective geometry we obtain:

Corollary 1 *Each linear mapping $\varphi : \mathcal{P}(\mathbf{V}) \rightarrow \mathcal{P}(\mathbf{V}')$ such that $\text{im } \varphi$ contains a triangle is induced by a semilinear mapping $f : \mathbf{V} \rightarrow \mathbf{V}'$ which is determined to within a non-zero factor in K' .*

The existence of a triangle in $\text{im } \varphi$ is needed in Corollary 1 to avoid “degenerate” cases:

1. If $\text{im } \varphi$ is a line, then such an f needs not exist, since a collineation of 1-dimensional projective spaces is just a bijection of their point sets. However, when $\#K \in \{2, 3, 4\}$, then φ is nevertheless induced by a semilinear mapping.
2. If $\text{im } \varphi$ is a singleton or empty, then K and K' need not be isomorphic. If K and K' are assumed to be isomorphic, then φ can be induced by a semilinear mapping.

We remark that under the assumptions of Corollary 1 the transpose of the semilinear mapping f induces the transpose of φ .

A geometric characterization of linear mappings of real projective spaces is due to REHBOCK [48], [49]. TIMMERMAN [54] characterizes the projections in projective spaces by conditions similar to our (L1). [9, 4.5] and a paper by FAURE and FRÖLICHER [21] contain proofs of Brauner's Theorem. A characterization of a linear mapping φ of a projective space in its dual space in terms of *two* mappings is given by FAURE and FRÖLICHER [23]. The two mappings are φ and $\varphi^T|_{\mathcal{P}}$, where \mathcal{P} is identified with a subspace of the bidual projective space. When these two mappings coincide, then one obtains a possibly degenerate quasipolarity. See also LENZ [34], [35].

1.4 Definition and Examples of Weak Linear Mappings

Formula (2) motivates the following definition: A *weak linear mapping* of projective spaces is a mapping $\varphi : \mathcal{P} \rightarrow \mathcal{P}'$ (defined on some subset of \mathcal{P}) satisfying

(WL1) $(\{X\} \vee \{Y\})^\varphi \subset \{X\}^\varphi \vee \{Y\}^\varphi$ for all $X, Y \in \mathcal{P}$, $X \neq Y$.

By (WL1), there are four possibilities for the image of a line $l \in \mathcal{L}$:

$\#(l \cap \text{ex } \varphi) \geq 2$: Then $l \subset \text{ex } \varphi$, i.e. we have an *exceptional line*.

$\#(l \cap \text{ex } \varphi) = 1$: Then all points of $l \cap \text{dom } \varphi$ are mapped to the same point.

$\#(l \cap \text{ex } \varphi) = 0$ and all points of l are mapped to the same point.

$\#(l \cap \text{ex } \varphi) = 0$ and $\varphi|_l$ is injective, but l^φ is not necessarily a line.

Thus now, in contrast to a linear mapping, we do not rule out the possibility that *all* points of a line are mapped onto the same point.

Each mapping $\varphi : \mathcal{P} \rightarrow \mathcal{P}'$, which is globally defined, injective, and collinearity-preserving, is a weak linear mapping. If non-collinearity of points is also preserved, then φ is called an *embedding*. An embedding is called *strong*, if independent points in $(\mathcal{P}, \mathcal{L})$ always go over to independent points in $(\mathcal{P}', \mathcal{L}')$.

We give some examples of weak linear mappings:

1. Formula (1) yields a weak linear mapping.
2. Each linear mapping is also a weak linear mapping.
3. Let L/K be a field extension and let \mathbf{V} be a vector space over K . Then

$$\varphi : \mathcal{P}(\mathbf{V}) \rightarrow \mathcal{P}(\mathbf{V} \otimes_K L), \mathbf{a}K \mapsto (\mathbf{a} \otimes 1)L$$

is a strong embedding. This φ is a collineation if, and only if, $K = L$ or $\dim \mathbf{V} \leq 1$.

4. The following embedding is not strong (BREZULEANU and RĂDULESCU [11]): Let L/K be a field extension such that there exist elements $1, y_0, y_1, y_2 \in L$ which are linearly independent in the *left* vector space L over K . Then define

$$\varphi : \mathcal{P}(K^4) \rightarrow \mathcal{P}(L^3), (a_0, a_1, a_2, a_3)K \mapsto (a_0 + y_0 a_3, a_1 + y_1 a_3, a_2 + y_2 a_3)L.$$

5. Each product of weak linear mappings is a weak linear mapping.

1.5 A Fundamental Theorem for Weak Linear Mappings

Theorem 2 (FAURE and FRÖLICHER [22], HAVLICEK [31]) *Each weak linear mapping $\varphi : \mathcal{P}(\mathbf{V}) \rightarrow \mathcal{P}(\mathbf{V}')$ such that $\text{im } \varphi$ contains a triangle is induced by a weak semilinear mapping $f : \mathbf{V} \rightarrow \mathbf{V}'$ which is determined to within a non-zero factor in K' .*

We infer from the decomposition of a weak semilinear mapping in 1.1 the following

Corollary 2 *Each weak linear mapping $\varphi : \mathcal{P} \rightarrow \mathcal{P}'$ of desarguesian projective spaces such that $\text{im } \varphi$ contains a triangle can be decomposed into a strong embedding $\mathcal{P} \rightarrow \mathcal{P}''$ in some projective space $(\mathcal{P}'', \mathcal{L}'')$ and a linear mapping $\mathcal{P}'' \rightarrow \mathcal{P}'$.*

Corollary 2 does not remain true if the hypothesis on $\text{im } \varphi$ is dropped: If $\text{im } \varphi$ does not contain a triangle of $(\mathcal{P}', \mathcal{L}')$, then $\text{ex } \varphi$ is still a subspace. For each point $Y \in \text{im } \varphi$ the extended preimage $\text{ex } \varphi \cup \{Y\}^{\varphi^{-1}}$ is a subspace. Thus $\mathcal{P}/\text{ex } \varphi$ is partitioned into subspaces. However, those subspaces are not necessarily isomorphic. Conversely, any subspace $\mathcal{S} \subset \mathcal{P}$ and any partition of \mathcal{P}/\mathcal{S} into subspaces determines a weak linear mapping in some projective line.

Corollary 3 *Let $\varphi : \mathcal{P}(\mathbf{V}) \rightarrow \mathcal{P}(\mathbf{V}')$ be given as in Theorem 2. Furthermore, assume that each monomorphism $K \rightarrow K'$ is surjective. Then φ is a linear mapping.*

For example, each monomorphism of \mathbb{R} is surjective [1, p. 88], whereas \mathbb{C} admits non-surjective monomorphisms [1, p. 114].

A recent result due to KREUZER [33] says that each pappian projective space is embeddable in a pappian projective plane. Other results on embeddings and specific examples are due to BENZ [1, p. 109], BREZULEANU and RĂDULESCU [11], BROWN [13], CARTER and VOGT [16], [17], HAVLICEK [30], and LIMBOS [37], [39], [40].

Another special class of weak linear mappings are *semicollineations* of projective spaces, i.e. globally defined bijective mappings which preserve the collinearity, but not necessarily the non-collinearity of points. CECCHERINI [18] has given an example of a semicollineation of a 4-dimensional projective space onto a non-desarguesian projective plane. Cf. also BERNARDI and TORRE [2], and MAROSCIA [43]. It seems to be an open problem, if each semicollineation of desarguesian projective spaces with dimensions ≥ 2 is a collineation.

1.6 Local Characterizations

The definition of a linear mapping φ uses all points of \mathcal{P} . The following “local” results characterize a linear mapping only in terms of its domain or in terms of a subset of its domain.

Firstly, we state the following variant of axiom (L1):

$$(L1') \quad (\{X\} \vee \{Y\})^\varphi = \{X\}^\varphi \vee \{Y\}^\varphi \text{ for all } X, Y \in \text{dom } \varphi, X \neq Y.$$

Theorem 3 (SÖRENSEN [52]) *Let $\varphi : \mathcal{P} \rightarrow \mathcal{P}'$ be a mapping of projective spaces satisfying (L1'). Then the following conditions hold true:*

1. *The image set $\text{im } \varphi \subset \mathcal{P}'$ is a subspace.*
2. *If $\text{span}(\text{ex } \varphi) \neq \mathcal{P}$ and if $\#\text{im } \varphi \geq 2$, then the exceptional subset $\text{ex } \varphi \subset \mathcal{P}$ is a subspace.*
3. *If $\text{span}(\text{ex } \varphi) \neq \mathcal{P}$ and if $\text{im } \varphi$ contains a triangle, then φ is a linear mapping.*

The subsequent examples illustrate that the assumptions in Theorem 3 are essential:

1. Let $\varepsilon : \mathcal{P}' \rightarrow \mathcal{P}$ be a non-surjective strong embedding with $\mathcal{P}' \neq \emptyset$, e.g., the canonical injection of a proper subspace $\mathcal{P}' \subset \mathcal{P}$. Define $\varphi : \mathcal{P} \rightarrow \mathcal{P}'$ as inverse mapping of ε with domain \mathcal{P}'^ε . This φ satisfies (L1'), but $\text{ex } \varphi \subset \mathcal{P}$ is not a subspace.
2. Let $\#\mathcal{P} > 1$ and $\#\mathcal{P}' = 1$. There exists an $\mathcal{M} \subset \mathcal{P}$ other than a subspace. Define $\varphi : \mathcal{P} \rightarrow \mathcal{P}'$ by $\text{ex } \varphi = \mathcal{M}$. Hence $\#\text{im } \varphi = 1$ and (L1') is trivially true.

3. Let $(\mathcal{P}, \mathcal{L})$ be a projective space containing a line t_1 with more than three points. Choose a complement \mathcal{S}_1 of t_1 and let \mathcal{S} be a hyperplane containing \mathcal{S}_1 . Write $\varphi_1 : \mathcal{P} \rightarrow t$ for the projection onto t with exceptional subspace \mathcal{S}_1 and put $t := t_1 \setminus \mathcal{S}$. The mapping $\varphi : \mathcal{P} \setminus \mathcal{S} \rightarrow t$, $X \mapsto X^{\varphi_1}$ satisfies (L1') with $(\mathcal{P}', \mathcal{L}') = (t, \{t\})$, but clearly (L1) is violated. Thus φ is not a linear mapping [52].

Secondly, axioms (L1) and (L2) are modified as follows:

- (L1'') X, Y, Z collinear implies $X^\varphi, Y^\varphi, Z^\varphi$ collinear, for all $X, Y, Z \in \text{dom } \varphi$.
(L2'') If $X^\varphi = Y^\varphi$ for distinct points $X, Y \in \text{dom } \varphi$, then $\varphi \upharpoonright ((\{X\} \vee \{Y\}) \cap \text{dom } \varphi)$ is a constant mapping.

Moreover, we need some topological tools: A projective space $(\mathcal{P}, \mathcal{L})$ is said to carry a *linear topology*, if each line $x \in \mathcal{L}$ is endowed with a non-trivial³ topology T_x such that all perspectivities between intersecting lines are continuous. A subset \mathcal{O} of \mathcal{P} is called *linearly open*, if $\mathcal{O} \cap x$ is open in the topological space (x, T_x) for all lines $x \in \mathcal{L}$ [3]. If $(\mathcal{P}, \mathcal{L})$ is a topological projective space [14, Ch. 23], then the induced topologies on the lines yield a linear topology. Each open set $\mathcal{M} \subset \mathcal{P}$ is also linearly open. However, there are linear topologies that do not arise in this way. An example is given by the *cofinite topology* in a projective space with infinite order: A subset m of a line a is defined to be open, if $a \setminus m$ is finite.

Theorem 4 (FRANK [24]) *Let $(\mathcal{P}, \mathcal{L})$ and $(\mathcal{P}', \mathcal{L}')$ be projective spaces satisfying the minor axiom of Desargues. Suppose that $\varphi : \mathcal{P} \rightarrow \mathcal{P}'$ is a mapping satisfying (L1'') and (L2'') such that $\text{im } \varphi$ contains a triangle. Each of the following conditions is sufficient for the existence of a unique linear mapping $\bar{\varphi} : \mathcal{P} \rightarrow \mathcal{P}'$ extending φ :*

- (a) *$(\mathcal{P}, \mathcal{L})$ admits a linear topology such that $\text{dom } \varphi$ is linearly open. Moreover, there exists a line $l \in \mathcal{L}$ such that $(\text{dom } \varphi \cap l)^\varphi$ contains non-empty open set with respect to some linear topology of $(\mathcal{P}', \mathcal{L}')$.*
(b) *$(\mathcal{P}, \mathcal{L})$ and $(\mathcal{P}', \mathcal{L}')$ are spaces of the same finite order N . Moreover, $x \cap \text{dom } \varphi \neq \emptyset$ implies $\#(x \cap \text{dom } \varphi) > \frac{2}{3}N$ for all $x \in \mathcal{L}$.*

Theorem 4 generalizes a result of LENZ [36] for real projective spaces. There are also several theorems in the literature giving local characterizations of embeddings of projective planes and spaces. Refer to BOGNÁR and KERTÉSZ [6], BREZULEANU and RĂDULESCU [12], and the references given in [24].

2 Other Spaces

2.1 Partial Linear Spaces and Linear Spaces

The geometric conditions (L1), (L2), (L1') etc. still do make sense in a wider context.

Let $(\mathcal{P}, \mathcal{L})$ be a pair consisting of a *point set* \mathcal{P} and a set $\mathcal{L} \subset 2^{\mathcal{P}}$ whose elements are called *lines*. If any two distinct points are on at most one common line, and if each line

³We rule out the coarsest and the finest topology.

contains at least two points, then $(\mathcal{P}, \mathcal{L})$ is called a *partial linear space*. A *linear space* is a partial linear space such that any two distinct points are on a common line.

SÖRENSEN [52] has discussed mappings between linear spaces $(\mathcal{P}, \mathcal{L})$ and $(\mathcal{P}', \mathcal{L}')$ satisfying (L1). Given $\mathcal{M} \subset \mathcal{P}$ the *trace space* $(\mathcal{M}, \mathcal{L}_{\mathcal{M}})$, which is defined by putting

$$\mathcal{L}_{\mathcal{M}} := \{l \cap \mathcal{M} \mid l \in \mathcal{L}, \#(l \cap \mathcal{M}) \geq 2\},$$

is also a linear space. Thus, if a linear mapping is not globally defined, one can go over to the trace space on its domain. Hence it means no loss of generality, that all mappings in [52] are assumed to be globally defined.

Examples of semicollineations of linear spaces and necessary conditions for a semicollineation of linear spaces to be a collineation are due to KREUZER [32]. There is a widespread literature on embeddings of linear spaces in projective spaces [14, Ch. 6].

2.2 Affine Spaces

The following seems to be part of the folklore: Suppose that a globally defined mapping φ of the affine space on \mathbf{V} (over K with $\#K > 2$) in the affine space on \mathbf{V}' is satisfying (L1). If $\text{im } \varphi$ contains a triangle, then $\varphi - 0^\varphi$ is a semilinear mapping $\mathbf{V} \rightarrow \mathbf{V}'$.

This result follows from Theorem 4 by going over to the projective closures and, for infinite order, by applying their cofinite topology. See SÖRENSEN [52] for a direct proof.

Globally defined mappings of affine spaces with property (WL1) have been discussed by ZICK [58], [59]. Since such a mapping needs not preserve parallelity of lines, *fractional weak semilinear mappings* have been introduced to obtain an algebraic description.

Even an *embedding* φ of the affine space on \mathbf{V} in the affine space on \mathbf{V}' has in general not the property that $\varphi - 0^\varphi$ is a weak semilinear mapping. One class of examples is given by spaces over $\text{GF}(2)$. A more interesting example is the embedding of an affine plane over $\text{GF}(3)$ in the complex affine plane which is based on the nine inflections of an elliptic cubic curve, another example is due to BENZ [1, p. 113]. See BICHARA and KORCHMÁROS [4], CARTER and VOGT [16], [17], LIMBOS [37], [38], [39], [40], OSTROM and SHERK [44], RIGBY [50], THAS [53]. SCHAEFFER [51] has given a local characterization of some embeddings which can also be found in [1, 3.2].

2.3 Grassmann Spaces

Write ${}^d\mathcal{P}$ for the set of all d -flats (i.e. d -dimensional subspaces) of an n -dimensional projective space $(\mathcal{P}, \mathcal{L})$ ($1 \leq d \leq n - 2$) and ${}^d\mathcal{L}$ for the set of all pencils of d -flats. Then $({}^d\mathcal{P}, {}^d\mathcal{L})$ is a partial linear space, called *Grassmann space*. Two distinct “points” ${}^dX, {}^dY \in {}^d\mathcal{P}$ are “collinear” if, and only if, ${}^dX \cap {}^dY$ is a $(d - 1)$ -flat. The Grassmann space $({}^d\mathcal{P}, {}^d\mathcal{L})$ is covered by one-dimensional projective spaces, namely the pencils of d -flats.

A mapping $\varphi : {}^d\mathcal{P} \rightarrow \mathcal{P}'$ in the point set of a projective space $(\mathcal{P}', \mathcal{L}')$ is called *linear*, if the restriction of φ to each pencil is a linear mapping of projective spaces [26].

If $(\mathcal{P}, \mathcal{L})$ is pappian, then each linear mapping $\varphi : {}^d\mathcal{P} \rightarrow \mathcal{P}'$ determines a mapping $\hat{\varphi} : G_{n,d} \rightarrow \mathcal{P}'$ of the associated Grassmann variety $G_{n,d}$ (cf. BURAU [15]) such that the

restriction to each line $l \subset G_{n,d}$ is linear. By HAVLICEK [27], this $\hat{\varphi}$ extends to a unique linear mapping of the ambient space of $G_{n,d}$. A weaker result is due to WELLS [55].

If $(\mathcal{P}, \mathcal{L})$ is not pappian, then Grassmann varieties are not available due to a non-existence theorem [27], and little seems to be known here.

A *geometric hyperplane* of $({}^d\mathcal{P}, {}^d\mathcal{L})$ is a subset of ${}^d\mathcal{P}$ which intersects each pencil of d -flats in exactly one or in all elements. The geometric hyperplanes are exactly the exceptional sets of linear mappings $\varphi : {}^d\mathcal{P} \rightarrow \mathcal{P}'$ with $\#\text{im } \varphi = 1$. Hence in the pappian case all geometric hyperplanes arise as hyperplane sections of the associated Grassmann variety [27, 4.5]. All geometric hyperplanes in the non-pappian case have been described by HALL and SHULT [25].

It seems that ECKHART [19] and REHBOCK [48], [49] have been the first geometers to discuss linear mappings of the Grassmann space formed by the lines of the real projective 3-space. See BRAUNER [7], [8], [10], HAVLICEK [28], and LÜBBERT [41] for further references and *kinematic line mappings*. ZANELLA [56] has investigated linear mappings of Grassmann spaces which are globally defined and injective. He has given sufficient conditions for the image of such a mapping to be projectively equivalent to the corresponding Grassmann variety and rather sophisticated examples where this is not the case. Cf. also BICHARA and ZANELLA [5].

2.4 Product Spaces

Another partial linear space is the *product space* of two projective spaces. If both spaces have isomorphic commutative ground fields, then their product space can be represented as a *Segre variety*. Linear mappings of a product space in a projective space may be defined as for Grassmann spaces. A thorough discussion of globally defined injective linear mappings can be found in a paper by ZANELLA [57]. Unfortunately, we cannot give the details here due to the lack of space. Let us just remark that the situation is more complicated than for Grassmann spaces. Cf. also BICHARA and ZANELLA [5].

2.5 Lattice Geometries

A significant generalization of linear mappings to *projective* and *affine lattice geometries* has been given by PFEIFFER [45], [46], and PFEIFFER and SCHMIDT [47]. See furthermore FAIGLE [20].

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